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OSCILLATIONS OF AN ELASTIC INHOMOGENEOUS STRIP CAUSED BY MOVING LOADS*

A.V. BELOKON'

There is studied the problem of the motion of a load on the boundary of an inhomogeneous elastic strip. It is assumed that the load moves at a constant sub-seismic velocity, and a regime of steady harmonic oscillations exists in the moving coordinate system. It is proved that the solution of the problem under consideration can be obtained easily if the solution of the "appropriate" problem of the oscillations of an inhomogeneous strip is known. Estimates are given of the velocity of the motion and the frequency of the oscillations at which there exists a unique energy solution of the problem. When the oscillations frequency is such that the energy solution does not exist, principles are formulated for extracting the unique solution.

1. Let a source of perturbations move at a constant velocity w along the boundary of an elastic medium. We shall study two kinds of problems, stationary and nonstationary. We call stationary the problem in which the state of stress and strain is independent of time in the coordinate system coupled to the moving perturbations source. Otherwise we shall say that we deal with a nonstationary problem.

Stationary problems were first studied in /1,2/. Contact stationary problems were considered in /3/ for homogeneous isotropic or pre-stressed strips. Stationary problems were studied in the monograph /4/ for load motion at a superseismic velocity over the boundary of a multilayered anisotropic foundation.

Problems with moving perturbations are studied separately in all the works listed, and without relationship to analogous problems about the harmonic oscillations of a strip. The relationship between these problems is made below and it is shown that on the basis of the "principles of correspondence" /5/, the properties of the solutions of the stationary and nonstationary problems can be obtained directly on the basis of investigating "corresponding" problems about strip oscillations /6 – 8/. Also analyzed here are the principles for selecting the unique solution. Motion to just the pre-seismic velocity is considered.

Let an elastic inhomogeneous anisotropic strip fill a domain S $(-\infty < x_1 < \infty;\ 0 \leqslant x_2 \leqslant 1)$. We will consider that we deal with an anisotropic material of general form and that the material constants depend only on x_2. In this case the stresses and strains are related by the Hooke's law relations

$$\sigma^{pj} = C^{pjkl}(x_2)\,\varepsilon_{kl};\quad p, j, k, l = 1, 2 \tag{1.1}$$

$$2\varepsilon_{kl} = u_{k,l} + u_{l,k} \tag{1.2}$$

Substituting (1.1) into the equations of motion, we find

$$(C^{pjkl}(x_2)\,u_{k,l})_{,j} = \rho(x_2)\,u_p^{\cdot\cdot} \tag{1.3}$$

Let a source of perturbations move at constant velocity w on the strip boundary. We shall henceforth assume that the boundary $x_2 = 0$ is clamped, while the following conditions are given on the boundary $x_2 = 1$

Problem A

$$\sigma^{2j}(x_1, 1, t) = f^j(x_1)\,e^{i\omega t};\quad f^j(x_1) = 0,\quad |x_1| > a;\quad j = 1, 2 \tag{1.4}$$

Problem B

$$\sigma^{2j}(x_1, 1, t) = f^j(x_1 - wt);\quad f^j(x) = 0,\quad |x| > a;\quad j = 1, 2 \tag{1.5}$$

Problem C

$$\sigma^{2j}(x_1, 1, t) = f^j(x_1 - wt)\,e^{i\Omega t};\quad f^j(x) = 0,\quad |x| > a;\quad j = 1, 2 \tag{1.6}$$

We call the problems A–C corresponding in the sense that in a moving coordinate system

$$x = x_1 - wt,\ y = x_2 \tag{1.7}$$

*Prikl.Matem.Mekhan.,46,No.2,pp.296-302,1982 the boundary conditions are identical and expressed by using the very same functions $f^j(x)$.

The following theorem is fundamental to a further study.

Theorem 1. Let the solution of the problem A be known

$$u^A = g(x_1, x_2, \omega) e^{i\omega t}$$

and let the Fourier transform of the vector-function g with respect to the coordinate x_1 have the form

$$\int_{-\infty}^{\infty} g(x_1, x_2, \omega) e^{i\gamma x_1} dx_1 = \frac{G(\gamma, x_2, \omega)}{D(\gamma, \omega)} \tag{1.8}$$

Then the solution of problem C is determined by the formula

$$u^c(x, y, t) = \frac{e^{i\Omega t}}{2\pi} \int_L \frac{G(\gamma, y, \Omega + \gamma w)}{D(\gamma, \Omega + \gamma w)} e^{i\gamma x} d\gamma \tag{1.9}$$

and the solution of problem B by (1.9), where we should set $\Omega = 0$.

For the proof we make a change of variables of the form (1.7) in the equations (1.3) and the boundary conditions (1.5), and we obtain

$$\begin{gathered} (C^{pjkl}(y) u_{k,l})_{,j} = \rho(y)(\ddot{u}_p + w^2 u_{p,11} = -2w\dot{u}_{p,1}) \\ \sigma^{2j}(x, 1, t) = f^j(x) e^{i\Omega t}, \ \ u(x, 0, t) = 0 \end{gathered} \tag{1.10}$$

We shall seek the solution of the boundary value problem (1.10) in the form

$$u(x, y, t) = v(x, y,) e^{i\Omega t} \tag{1.11}$$

Substituting (1.11) into (1.10) and applying the Fourier transform, we find

$$\begin{gathered} L(V) + \beta\rho(y) V = 0; \ M(V) = F, \ y = 1; \ V = 0, \ y = 0 \\ V = \int_{-\infty}^{\infty} v(x, y) e^{i\gamma x} dx; \quad F^j = \int_{-a}^{a} f^j(x) e^{i\gamma x} dx \\ L(V) = L_0(V) - i\gamma L_1(V) - \gamma^2 L_2(V) \\ M(V) = M_0(V) - i\gamma M_1(V); \ \beta = (\Omega + \gamma w)^2 \end{gathered} \tag{1.12}$$

The differential expressions of the two-dimensional vectors L_p and M_p are determined by using the relations

$$\begin{gathered} L_{0j} = (C^{j2k2}(y) V_{k,2})_{,2}, \quad L_{1j} = C^{j1k2}(y) V_{k,2} + (C^{j21k}(y) V_k)_{,2} \\ L_{2j} = C^{j1 1k}(y) V_k, \quad M_{0j} = C^{j2k2}(y) V_{k,2}, \quad M_{1j} = C^{2jk1}(y) V_k \end{gathered}$$

Now, if we seek the solution of problem A in the form (1.11), then after analogous manipulations we will arrive at a boundary value problem of the form (1.12), only we should set $\beta = \omega^2$, $\omega = \Omega$. From a comparison of the boundary value problems A and C obtained in the Fourier transforms, theorem 1 results. The question of the selection of the contour of integration L in (1.9) will be discussed below.

Theorem 1 permits investigating the problems C and B on the basis of the properties of the solution to problem A which have already been studied /6 – 8/.

2. A study of the zeroes of the dispersion equation

$$D(\gamma, \Omega + \gamma w) = 0 \tag{2.1}$$

will be important for the sequel.

Here we devote the main attention just to the real projection of the zeroes of the dispersion equation. We let Γ_1 denote the set of points in a plane of two real variables (γ, ω) that satisfy the equation

$$D(\gamma, \omega) = 0 \tag{2.2}$$

Lemma 1. The set of points Γ_2 satisfying (2.1) in the plane of two real variables γ, Ω ($w \geqslant 0$ is fixed) consists of points of intersection of the line

$$\omega = \Omega + \gamma w \tag{2.3}$$

with the set Γ_1.

Lemma 1 is obvious. Therefore, the points belonging to Γ_2 can be found by means of the known set Γ_1 by a graphical construction (see Fig.1).

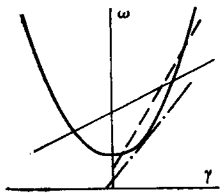

Fig.1

Equation (2.1) can have zeroes with different signs (solid line), with the identical signs (dashed line), and multiple zeroes (dash-dot line). We call the first two types of roots regular, and the last irregular. We show below that different mechanical phenomena occurring in the strip correspond to these three types of roots.

The sequent lemmas follow from results obtained earlier /6,7/.

Lemma 2. Positive constants m, ω_0 exist such that upon compliance with the conditions

$$w < (1 - \Omega/\omega_0)\sqrt{m/\rho_0},\ \Omega < \omega_0,\ \rho_0 = \max_{0\leqslant y\leqslant 1} \rho(y) \tag{2.4}$$

equation (2.1) will have no real roots. When conditions (2.4) are violated, the equation can have real roots, and always a finite number if only

$$w < \sqrt{m/\rho_0},\ \ \Omega < M = \mathrm{const} < \infty \tag{2.5}$$

Lemma 3. Let $V_k^\circ(y)$ be an eigenfunction, the solution of the homogeneous boundary value problem (1.12) for $F = 0$ and $\gamma_k, \Omega \in \Gamma_2$ and let $V_k^1, V_k^2, \ldots, V_k^q$ be vector functions associate to V_k°. In this case the solution of the homogeneous problem (1.10), (1.11) will have the form

$$u_k^q = (\exp(\Omega t - \gamma_k x)\, i) \sum_{s=0}^{q} \frac{(-ix)^s}{s!} V_k^{q-s} \tag{2.6}$$

3. We study questions associated with the existence of energy solutions of problems B and C. For later, we introduce the scalar product

$$(a, b)_{H_{1s}} = \int_S C^{pjkl}(y)\, \varepsilon_{pj}(a)\, \overline{\varepsilon_{kl}(b)}\, dS \tag{3.1}$$

$$(a, b)_{W_2^1} = \int_S \sum_{k,\, j=1}^{2} a_{k,\, j} \overline{b_{k,\, j}}\, dS$$

in the set H of vector-functions a continuous with their first derivatives and taking the value zero at $y = 0$.

We call the spaces H_{1s} and W_2^1, respectively, the closure of the set H or of its corresponding subsets in norms corresponding to (3.1). Let us also introduce the space H_s

$$(a, b) = \int_S \rho(y)\, a \cdot \bar{b}\, dS \tag{3.2}$$

The following inequalities hold /8/:

$$\omega_0^2 \| a \|_{H_s}^2 \leqslant \| a \|_{H_{1s}}^2; \quad m \| a \|_{W_2^1}^2 \leqslant \| a \|_{H_{1s}}^2 \leqslant m_1 \| a \|_{W_2^1}^2 \tag{3.3}$$

where ω_0 and m are constants in (2.4)

We call the vector function $u \in H_{1s}$ satisfying the integral indentities

$$(u, a)_{H_{1s}} - \int_S \rho(y)\, w^2 u_{k,1} a_{k,1}\, dS = \int_{-a}^{a} f^k(x)\, a_k(x, 1)\, dx \tag{3.4}$$

$$(u, a)_{H_{1s}} - \int_S [w^2 u_{k,1} \overline{a_{k,1}} + iw\Omega(u_{k,1} \overline{a_k} - u_k \overline{a_{k,1}}) + \Omega^2 u_k \overline{a_k}]\, \rho(y)\, dS = \int_{-a}^{a} f^k(x)\, \overline{a_k(x, 1)}\, dx \tag{3.5}$$

respectively, for arbitrary $a \in H_{1s}$ the generalized solution of problems B and C.

Theorem 2. Let conditions (2.4) be satisfied and $f^j(x) \in L_p(-a, a), p > 1$. In this case a unique generalized solution of problem C (and of problem B) exists in H_{1s}.

Indeed, by estimating the integral in the left side of (3.5) while taking (3.3) into account, we obtain

$$\int_S \rho(y)\, [w^2 u_{k,1} \overline{a_{k,1}} + iw\Omega(u_{k,1} \overline{a_k} - u_k \overline{a_{k,1}}) + \Omega^2 u_k \overline{a_k}]\, dS \leqslant (\sqrt{\rho_0/m} + \Omega/\omega_0)^2 \| u \|_{H_{1s}} \| a \|_{H_{1s}}$$

and since condition (2.4) is satisfied, then an auxiliary space H_{1s}° can be introduced with a scalar product defined by the left side of the integral identity (3.5). The space introduced and H_{1s} are again equivalent. Furthermore, there follows from the conditions of Theorem 2 and the imbedding theorem /9/

$$\left| \int_{-a}^{a} f^k(x) \overline{a_k(x, 1)}\, dx \right| \leqslant m_1 \| f \|_{L_p} \| a \|_{H_{1s}}$$

Hence, because of the equivalence of H_{1s} and $H_{1s}{}^{\circ}$, unique solvability of the problem C (and of problem B) results. Here

$$\| u \|_{H_{1s}} \leqslant m \| f \|_{L_p}$$

4. The question of selecting the unique solution occurs in studying steady oscillations of a strip in the case conditions (2.4) are not satisfied.

The Ignatovskii /10/ principle of ultimate absorption, the Mandel'shtam /11/ principle of energetic radiation, the Sommerfeld principle /10/, and also the Tikhonov – Samarskii principle /10/ of ultimate amplitude are usually used in problem A. A complete analysis of the first three principles for problem A is given in /6,7/.

We analyze the situation that occurs in problem C. We assume that the roots of equation (2.1) are simple. The homogeneous solution of the problem C then has the form (2.6) or in real form

$$u(x, y, t) = v_s(y) \sin \delta + v_c(y) \cos \delta, \ \delta = \Omega t - \gamma x \tag{4.1}$$

Let us obtain the energy relations. To do this we miltiply the first equation of the system (1.10) by $u_1^{\cdot}$, the second by $u_2^{\cdot}$ and add. Integrating the result with respect to y between 0 and 1, we find

$$\frac{\partial \mathrm{E}}{\partial t} + \frac{\partial J}{\partial x} = 0 \tag{4.2}$$

It was taken into account in the derivation of (4.2) that the solution (4.1) satisfies homogeneous boundary conditions, and moreover, the energy E and the energy flux J through a section $x = \mathrm{const}$ are determined by the formulas

$$2\mathrm{E} = \int_0^1 \{C^{pjkl}(y) \varepsilon_{pj}(u) \varepsilon_{kl}(u) + \rho(y) [u_1^{\cdot 2} + u_2^{\cdot 2} - w^2 (u_{1,1}^2 + u_{2,1}^2)]\} \, dy$$

$$J = -\int_0^1 [\sigma^{1k} u_k^{\cdot} + w\rho(y) (u_1^{\cdot 2} + u_2^{\cdot 2} - w u_{k,1} u_k^{\cdot})] \, dy$$

Using the Hooke's law relationship and (4.1), as well as the equation of motion, we obtain

$$A(\sigma(v), a) + \beta B(v, a) = 0 \tag{4.3}$$

$$A(\sigma(v), a) = \int_0^1 [\gamma \sigma_c^{1k}(v) a_{sk} - \sigma_s^{2k}(v) a_{sk,2} - \sigma_c^{2k}(v) a_{ck,2} - \gamma \sigma_s^{1k}(v) a_{ck}] \, dy$$

$$B(v, a) = \int_0^1 \rho(y) (v_s \cdot a_s + v_c \cdot a_c) \, dy$$

Now setting $a = v$ in (4.3), we differentiate the equality obtained with respect to γ. We then find by taking account of (4.3) that

$$\frac{d\beta}{d\gamma} B(v, v) + 2C(\sigma(v), v) = 0 \tag{4.4}$$

$$C(\sigma(v), v) = \int_0^1 [\sigma_c^{1k}(v) v_{sk} - \sigma_s^{1k}(v) v_{ck}] \, dy$$

The subscripts c and s mean that the appropriate function is represented in a form analogous to (4.1). Finally, taking into account that

$$J_1 = \frac{1}{T} \int_0^T \int_0^1 \sigma^{1k}(u) u_k^{\cdot} \, dy \, dt = -\frac{\Omega}{2} C(\sigma(v), v); \qquad T = \frac{2\pi}{\Omega}$$

we obtain by taking account of (4.4)

$$J_1 = \frac{\Omega}{4} \frac{d\beta}{d\gamma} B(v, v) \tag{4.5}$$

A formula analogous to (4.5) was obtained by another method /7/.

Let us form a relationship for the energy flux in problem C. By using (4.5) and (4.1), we can establish that the energy flux through a section $x = \mathrm{const}$, calculated in a moving coordinate system and averaged over the period of the oscillations, is given by the relationship

$$P = \frac{1}{T}\int_0^T J\,dt = \frac{\Omega}{2}(\Omega + \gamma w)(c_g - w)\,B(v, v) \tag{4.6}$$

Here c_g is the group velocity of wave propagation in problem A, calculated at the point $(\gamma, \Omega + \gamma w)$ on the (γ, ω) plane.

Finally, we present the relationship for the energy averaged over the period of oscillations and calculated in a moving coordinate system

$$E = \frac{1}{T}\int_0^T \mathrm{E}\,dt = \frac{1}{2}\,\Omega(\Omega + \gamma w)\,B(v, v) \tag{4.7}$$

Comparing (4.6) and (4.7) we find

$$P = (c_g - w)E$$

Definition 1. By analogy with the problem A we consider those solutions to have a physical meaning, which denote energy transfer from the source to infinity (Mandel'shtam principle). In other words, on the basis of this definition, solutions are selected for which

$$p > 0,\ x \gg a;\ p < 0,\ x \ll a$$

Now, let us consider an elastic medium possessing little friction. This is equivalent to adding the term

$$2\varepsilon\rho(x_2)u_p^{\cdot}$$

to the right side of (1.3), where ε is an arbitrarily small number. We call the problem C for a medium with friction C_ε.

Lemma 4. Let the Fourier transformed solution of the problem C be known

$$V(\gamma, y, \Omega + \gamma w)\,e^{i\Omega t}$$

Then the Fourier transformed solution of the problem C_ε will have the form

$$V(\gamma, y, \Omega + \gamma w - i\varepsilon)e^{i\Omega t}$$

In fact, for a medium with little friction the right side of (1.3) can be written as

$$-\rho(y)\left[(\Omega + \gamma w)^2 - 2i\varepsilon(\Omega + \gamma w)\right]$$

after passing to a moving coordinate system, applying the Fourier transform, and dividing out the time. There hence results that to obtain the Fourier transformed solution of the problem C_ε in terms of the Fourier transformed solution of the problem C we should replace $\Omega + \gamma w$ by

$$\left[(\Omega + \gamma w)^2 - 2i\varepsilon(\Omega + \gamma w)\right]^{1/2} \approx \Omega + \gamma w - i\varepsilon$$

which, indeed, proves Lemma 4.

Theorem 3. Let an elastic medium possess little friction. In this case the equation

$$D(\gamma, \gamma w + \Omega - i\varepsilon) = 0 \tag{4.8}$$

has just complex roots. The following asymptotic formula

$$\gamma = \gamma_0 - i\varepsilon(c_g - w)^{-1} + o(\varepsilon) \tag{4.9}$$

holds for the regular roots of this equation, where γ_0 is a real root of (2.1). The first part of the theorem follows from the results obtained in /6/, where it is shown that if ω is a complex number in (2.2), then (2.2) has only complex roots. Let us prove the second part. We represent (4.8) in the form

$$-i\varepsilon + \Omega + \gamma w = \psi(\gamma) \tag{4.10}$$

where $\psi(\gamma)$ is an analytic function /8/, and we will seek the roots of (4.10) in the form $\gamma = \gamma_0 + i\lambda$. Expanding $\psi(\gamma)$ in a series in the neighborhood of γ_0, we arrive at (4.9), which indeed proves Theorem 3.

Definition 2. We call the limit of the solution of problem C_ε as $\varepsilon \to 0$ the solution of the problem C (the ultimate absorption principle of Ignatovskii).

On the basis of Definition 2 and Theorem 3 it can now be shown that in the case of regular roots for (2.1) the contour of integration L in (1.9) passes over the real axis γ, deviating from it only in the neighborhood of points that are zeroes of the function (2.1). Here if $c_g < w$, then the contour deflects upward, and downward otherwise.

Theorem 4. Let two solutions of the problem C be constructed that satisfy Definitions 1 and 2, respectively, then in the case of regular roots for the equation (2.1) and upon compliance with conditions (2.5) of Lemma 2, these solutions agree.

The proof of Theorem 4 can be executed by analogy with Theorem 19.1 in /8/.

5. Let us analyze the solution of problem C for those combinations of the parameters w and Ω for which equation (3.1) has only two real roots. We first consider the case of zeroes with different signs (the solid line in the Fig.1). Evaluating the integral (1.9) by residue theory, we find that the solution (1.9) contains two waves undamped at infinity, of the form

$$A_1 \exp [i(\Omega t - \gamma_1 x)],\ x > a;\ A_2 \exp [i(\Omega t - \gamma_2 x)],\ x < -a \tag{5.1}$$

Since $\gamma_1 > 0$ and $\gamma_2 < 0$, then both waves depart to infinity from the source of oscillations. If both roots are of identical sign (the dashed line in the Fig.1) $\gamma_1, \gamma_2 > 0$, then the first undamped wave (5.1) will, as before, depart from the source of oscillations while the second wave will arrive from $-\infty$ at the source of oscillations. Since in this case $c_g < w$, then on the basis of (4.6) the energy flux will be propagated from the source to infinity. The situation occurring when the wave is propagated in one direction but carries energy in the other direction holds even in problem A /8/. Let us also note that if $\gamma_2 = 0$ but $\gamma_1 > 0$ (this is possible if the vibration frequency Ω is such that it coincides with the beginning of the dispersion curve of problem A), then in this case it is generally meaningless to speak about the second wave in (5.1).

Finally, we consider the case when $\gamma_1 = \gamma_2$ (the dash-dot line in the Fig.1). In this case $c_g = w$ and (4.9) is not applicable. Proceeding exactly as in deriving (4.9), except retaining higher order terms in λ in the series expansion of $\psi(\gamma)$, we find

$$\lambda_{1,2} = \pm e^{\pi i/4} \sqrt{\varepsilon}\, (\partial c_g / \partial \gamma)^{-1}$$

It is seen that the real double pole is bifurcated in a medium with friction. Hence, if the ultimate absorption principle is used in constructing the solution (the Mandel'shtam principle is not applicable here) then in studying the problem C_ε we will have a simple pole but we do not obtain a uniform passage to the limit as $\varepsilon \to 0$. In this case as $\varepsilon \to 0$ the amplitudes in the solution of the form (5.1) will grow without limit, i.e., the behavior of the solution in a strip at a frequency for which $c_g = w$ is analogous to the behavior of the solution in a bounded body at the resonance frequency.

Therefore, the fundamental cases which can hold as a load moves over an inhomogeneous strip have been examined here. The case when (2.1) has a zero of order higher than the second can be examined analogously.

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